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AG II - Schemes and sheaves

Exercises – week 3

Exercise 1. *Nilradical.* Let R be a ring. Denote by

$$\mathrm{nil}(R) := \{f \in R \mid f \text{ is nilpotent}\}.$$

(1) Show that
$$\mathrm{nil}(R) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}.$$

(2) Show that for an ideal $I \subset R$, we have $V(I) = \mathrm{Spec}(R)$ if and only if every element of I is nilpotent, meaning $I \subset \mathrm{nil}(R)$.

Exercise 2. Spec *is an adjoint.* Let $(X, \mathcal{O}_X)$ be a scheme and A a ring. Show that the induced map on global sections

$$\mathrm{Hom}_{\mathrm{Sch}}((X, \mathcal{O}_X), \mathrm{Spec}(A)) \to \mathrm{Hom}_{\mathrm{Ring}}(A, \mathcal{O}_X(X))$$

is bijective. This implies that

$$\mathrm{Spec} \colon \mathrm{Ring}^{op} \to \mathrm{Sch}$$

is a right adjoint. In particular colimits of rings are sent to limits of schemes.

Remark. The above remains true if we replace Sch by the category of locally ringed spaces $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ring}}$. This characterizes Spec as the right adjoint of the global sections functor $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{Ring}} \to \mathrm{Ring}^{op}$. This formalize the saying that $\mathrm{Spec}(R)$ is the universal (locally ringed) space such that R is the ring of global functions on this space.

Exercise 3. *Reduced schemes.* A scheme $(X, \mathcal{O}_X)$ is *reduced* if for all opens U of X the ring $\mathcal{O}_X(U)$ is reduced.

(1) Show that a scheme $(X, \mathcal{O}_X)$ is reduced if and only if for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a reduced ring.
(2) Show that an affine scheme $\mathrm{Spec}(A)$ is reduced if and only if A is a reduced ring.

The *reduction* of a scheme X is a scheme X_{red} together with a map $\iota : X_{red} \to X$ with the property that for every map $Y \to X$ where Y is a reduced scheme, then Y factors uniquely to ι.

(3) Show that if $X = \mathrm{Spec}(A)$ then $\mathrm{Spec}(A/\mathrm{nil}(A)) \to \mathrm{Spec}(A)$ is the reduction of $\mathrm{Spec}(A)$.
(4) Show that the reduction of any scheme exists and that $\iota \colon X_{red} \to X$ is a homeomorphism.

Exercise 4. *Residue fields and rational points.* Let $(X, \mathcal{O}_X)$ be a scheme, $x \in X$ and $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ the *residue field at x.*

(1) Let K be a field. Show that a map $\mathrm{Spec}(K) \to X$ with topological image x amounts to a field extension $k(x) \to K$.
(2) Let k be a field. Fix $X \to \mathrm{Spec}(k)$ a map for the rest of the exercise. Show that for all $x \in X$, $k(x)$ is naturally a field extension of k.
(3) We say that $x \in X$ is *k-rational* if the natural extension of last item $k \to k(x)$ is an isomorphism. Show that the set of k-rational points of X is identified with the set of maps $\mathrm{Spec}(k) \to X$ such that the composite $\mathrm{Spec}(k) \to X \to \mathrm{Spec}(k)$ is the identity.
(4) Let now $X = \mathrm{Spec}(k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)) \to \mathrm{Spec}(k)$[1], where $f_1, \ldots, f_m$ are polynomials. Show that the set of k-rational points of X is identified with the set of solutions in k^n of the system of polynomials $f_1, \ldots, f_m$.

Exercise 5. *Exceptional functors (1).* Let X be a topological space. Let $j \colon U \to X$ be an open subset and $\iota \colon Z \to X$ its closed complement. We work with categories of sheaves of abelian groups on these spaces.

(1) Consider $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Z)$. Compute every stalk of $\iota_* \mathcal{F}$.
(2) Show that ι_* is exact.
(3) Give an example to show that j_* is not exact.

Consider $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(U)$. We define the *extension by zero* or *exceptional direct image* $j_! \mathcal{G}$ to be the sheafification of the presheaf defined by $V \mapsto \mathcal{G}(V)$ if $V \subset U$ and 0 otherwise.

(4) Show that for every sheaf $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$ there is a natural exact sequence
$$0 \to j_! j^{-1} \mathcal{H} \to \mathcal{H} \to \iota_* \iota^{-1} \mathcal{H} \to 0.$$
(5) Show that there is a natural bijection in $\mathcal{G} \in \mathrm{Sh}_{\mathrm{Ab}}(U)$ and $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$
$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(U)}(\mathcal{G}, j^{-1}\mathcal{H}) \cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(j_! \mathcal{G}, \mathcal{H}).$$
This means that for an open immersion j, we have a sequence of adjoints $j_! \dashv j^{-1} \dashv j_*$.

Exercise 6. *Exceptional functors (2).* We keep setup and notation as in previous exercise. Let $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$.

(1) Show that for every $s \in \mathcal{H}(V)$ for an open V, then
$$\mathrm{supp}(s) := \{x \in V \mid s_x \neq 0\}$$
is closed.
(2) Show that $\mathcal{H}_Z$, the presheaf on X defined by
$$\mathcal{H}_Z(V) = \{s \in \mathcal{H}(V) \mid \mathrm{supp}(s) \subset Z \cap V\}$$
is a sheaf. Show that $\mathcal{H}_Z(V)$ is the kernel of the map
$$\mathcal{H}(V) \to \mathcal{H}(V \cap (X \setminus Z)).$$

[1] Induced by the inclusion $k \to k[x_1, \ldots, x_n]$

(3) Show that if $V' \subset V$ such that $V' \cap Z = V \cap Z$ then the restriction map $\mathcal{H}_Z(V) \to \mathcal{H}_Z(V')$ is an isomorphism.

(4) Show that for any sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Z)$ any map $\iota_*\mathcal{F} \to \mathcal{H}$ factors through $\mathcal{H}_Z$.

We define the *exceptional inverse image* $\iota^!\mathcal{H} := \iota^{-1}\mathcal{H}_Z$.

(5) Show that there is a natural bijection in $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(Z)$ and $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(X)$

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(Z)}(\mathcal{F}, \iota^!\mathcal{H}) \cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}(X)}(\iota_*\mathcal{F}, \mathcal{H}).$$

This means that for a closed immersion ι, we have a sequence of adjoints $\iota^{-1} \dashv \iota_* \dashv \iota^!$.

Exercise 7. *Topological properties of schemes.* A topological space X is T_0 if for every pair of different elements $x, y \in X$ there exist an open set U of X such that exactly x or y is in U.

(1) Let X be the underlying topological space of a scheme. Show that X is T_0.

A topological space is called *irreducible* if it cannot be written as the union of two proper and non-empty closed subsets.

(1) Show that any non-empty open set of an irreducible topological space is dense.

(2) Show that if an irreducible topological space X contains at least two points, then X is not Hausdorff.

(3) Let A be a ring. Show that the topological space $\mathrm{Spec}(A)$ is irreducible if and only if A_{red} is an integral domain.

A topological space is called *sober* if for any non-empty irreducible closed subset $Z \subset X$, there exist a unique point $\eta_Z \in Z$ such that $\overline{\{\eta_Z\}} = Z$. In this case, we call η_Z the *generic point* of Z.

(1) Show that any Hausdorff topological space is sober.

(2) Let X be the underlying topological space of a scheme. Show that X is sober.

(3) Let A be an integral domain. What is the generic point of $\mathrm{Spec}(A)$?